Scale Independent Linear Behavior of Alluvial Channel Flow

He Qing Huang1 and Howard H. Chang, M.ASCE2

Abstract: For flow in a rigid open channel with no bed sediment, the achievement of the special state of stationary equilibrium yields a linear characteristic. To examine the existence of a linear characteristic in alluvial channel flow, this study presents a direct formulation of the special equilibrium state following a variational approach. It finds that a linear relationship between shear stress and width/depth ratio of alluvial channels emerges under the commonly identified flow resistance and sediment transport conditions. Most importantly, this linear relationship yields not only the theoretical equilibrium channel geometry that is very close to a widely identified empirical counterpart but also a nondimensional number $H$, defined as the ratio of the relative increment of shear stress to the increment of width/depth ratio. This study suggests that $H$ needs to be adopted as a criterion of hydraulic similitude for modeling sediment (bed-load) transport in alluvial channels.


CE Database subject headings: Open channel flow; Alluvial channels; Sediment transport.

Introduction

Fluid motion is governed by the nonlinear Navier-Stokes equations, with which a nonlinear system gains a unique solution only when it reaches a special equilibrium state. The Navier-Stokes law also governs open channel flow. A unique solution of channel geometry appears when the energy of flow reaches the minimum level for achieving equilibrium. In an analogy to the dynamic motion of solid materials, Huang et al. (2004a) define this special equilibrium state in the dynamic motion of fluids as “stationary equilibrium.” For flow in a frictionless, fixed-boundary open channel, the stationary equilibrium state defines the critical flow depth, at which the following linear characteristic is exhibited (e.g., Henderson 1966):

$$E_p = 2E_k$$

(1)

where $E_p$ and $E_k$ are specific potential energy and kinetic energy of flow, respectively.

Letting $E_p$ and $E_k$ be represented, respectively, with $h$ and $V^2/2g$ ($h = \text{vertical distance from channel bed to water surface, or flow depth}$; $V = \text{average velocity of flow}$; and $g = \text{acceleration due to gravity}$), Eq. (1) leads the Froude number $F$ to unity

$$F = \frac{V}{\sqrt{gh}} = 1$$

(2)

Because in a frictionless fixed-boundary open channel, flow spends energy in the forms of potential and kinetic energy, Eq. (1) or Eq. (2) physically means that flow achieves stationary equilibrium only when the ratio between the two types of energy remains at an appropriate level. When the ratio cannot be maintained due to an excessive supply of energy, flow will be in a dynamic equilibrium form, in which the condition presented in Eq. (1) or Eq. (2) plays a role as an internal control (Chow 1959; Henderson 1966; Huang et al. 2004a).

For flow in a frictional nonadjustable open channel, the stationary equilibrium state yields the best hydraulic section, which has the following linear characteristic (e.g., Chow 1959; Henderson 1966; Huang et al. 2004a):

$$\zeta = \rho$$

(3)

in which $\zeta = \text{channel shape factor normally represented with the width/depth ratio of cross section and $\rho =$ constant.}$ When channels take a rectangular or a semicircular or a triangular cross section, $p$ takes a value of 2.0 or 2.546 or 4.0, correspondingly (Chow 1959).

In a frictional fixed-boundary open channel, flow expends energy to overcome friction from channel boundary. Eq. (3) physically means that to attain stationary equilibrium, the wetted perimeter (channel boundary) is a minimum for a given cross-sectional area. In this case, the channel shape is neither very wide and shallow nor very narrow and deep, so that energy expenditure achieves a minimum (Huang et al. 2004a).

For flow in alluvial channels, however, it is unknown whether and on what conditions a linear characteristic exists when flow achieves stationary equilibrium. Nevertheless, the stationary equilibrium state of flow in all types of open channel is governed by the same principle—that of minimum energy (expenditure), or the principle of least action in the origin (Huang et al. 2004a,b). It is of interest to determine if linearity can be derived from the same principle for alluvial channels.

For this purpose, this study presents a direct formulation of the stationary equilibrium state for alluvial channel flow following a variational approach. To account for the effects of sediment transport, this study adopts a generalized bed-load transport formula.
that is based on the shear stress approach. It is then found that when alluvial channel flow achieves stationary equilibrium, a linear relationship between shear stress and width/depth ratio emerges under the commonly identified sediment transport conditions. It will be shown that the theoretically derived equilibrium alluvial channel geometry under the linear condition is very close to widely identified empirical hydraulic geometry. Most importantly, this study identifies a nondimensional number \( H \) from the linear relationship. By analyzing the physics underlying \( H \), this study suggests that \( H \) controls the behavior of alluvial channel flow and is an important criterion of hydraulic similarity for modeling bed-load transport in alluvial channels.

**Variational Formulation of Stationary Equilibrium State**

For imposed water and sediment loads, alluvial channel flow achieves equilibrium when the following conditions are satisfied:

\[
Q_c = Q \quad Q_{sc} = Q_s
\]

where \( Q_c \), \( Q_s \), and \( Q_{sc} \) are carrying capacity of flow for water discharge, the water discharge imposed to flow, the carrying capacity of flow for sediment discharge, and the sediment discharge imposed to flow, respectively.

Basic relationships of flow continuity, resistance, and sediment transport have previously been seen to be insufficient to explain the self-adjusting mechanism of alluvial channels and as a consequence several extremal hypotheses have been proposed. These include maximum sediment transporting capacity (MSTC) (e.g., Kirkby 1977; White et al. 1982), minimum stream power (MSP) (e.g., Chang 1979a,b, 1980a,b, 1985, 1988; Millar and Quick 1993, 1998), maximum friction factor (Davis and Sutherland 1980, 1983), minimum energy dissipation rate (Yang et al. 1981), and minimum Froude number (Yalin and Silva 1999, 2000). These extremal hypotheses, however, lack convincing physical justification (e.g., Knighton 1998).

Without predestining any extremal hypothesis, nevertheless, the recent studies of Huang and co-workers (e.g., Huang and Nanson 2000, 2001, 2002; Huang et al. 2002, 2004a,b) have demonstrated that the physical mechanism governing the adjustment of alluvial channel cross sections can be explained directly with the basic relationships of flow continuity, resistance, and sediment (bed-load) transport. These studies have revealed three important findings. First, they show the necessity of reducing the number of independent variables through introducing a nondimensional channel shape factor \( \zeta \) into detailed mathematical analyses. Second, they demonstrate that among numerous forms of equilibrium channel geometry, unique channel geometry occurs when the efficiency of flow in transporting sediment load reaches a maximum. This unique section is defined as the most efficient alluvial channel cross section that is based on the shear stress approach. It is then found that when alluvial channel flow achieves stationary equilibrium, a linear relationship between shear stress and width/depth ratio emerges under the commonly identified sediment transport conditions. It will be shown that the theoretically derived equilibrium alluvial channel geometry under the linear condition is very close to widely identified empirical hydraulic geometry. Most importantly, this study identifies a nondimensional number \( H \) from the linear relationship. By analyzing the physics underlying \( H \), this study suggests that \( H \) controls the behavior of alluvial channel flow and is an important criterion of hydraulic similarity for modeling bed-load transport in alluvial channels.

In these findings, two alternative approaches for formulating the stationary state of equilibrium for flow in alluvial channels are suggested: (1) minimizing energy gradient for given water discharge and sediment load; and (2) maximizing sediment (bed-load) transporting capacity for given water discharge and energy gradient. In comparison, the second approach involves much straightforward and simpler mathematical analyses and so is adopted here in the following form:

\[
(Q_{sc})_{\max} = \max_{Q_s}(Q_s(\zeta)|Q_s(Q) = Q_s) = Q_s
\]

where \( (Q_{sc})_{\max} = \max_{Q_s} \) maximum sediment carrying capacity of flow; and \( \zeta \) = nondimensional shape factor of channel cross section.

To find the solutions of Eq. (5), the following variational equation against \( \zeta \) must be solved:

\[
\frac{dQ_{sc}}{d\zeta} \bigg|_{dQ_s/d\zeta = 0} = 0
\]

Because it is unclear what a complex profile of channel section natural streams can take at this stage, we regard the profile as a function with uncertainty in this study. Within the uncertainty, nevertheless, channel width is normally measurable and so can be regarded as a major independent variable. For these reasons, the variation of the shape of alluvial channel cross section is roughly illustrated in the following form:

\[
d\zeta = \frac{B^2}{A}
\]

where \( A \) and \( B \) = channel cross-sectional area and width, respectively.

Letting \( h \) be the ratio of \( A/B \), or the averaged channel depth, Eq. (7) leads to the following variational relationships:

\[
\frac{1}{B} \frac{d}{d\zeta} = \frac{1}{\zeta} \frac{1}{h} \frac{dh}{d\zeta} = \frac{1}{A} \frac{1}{d\zeta} \frac{dA}{d\zeta} + \frac{2}{h} \frac{dh}{d\zeta}
\]

where \( R \) = hydraulic radius of channel cross section.

Because the carrying capacity of flow for water discharge is illustrated with the relationship of \( Q_c = AV \), the following variational relationship needs to be satisfied for flow to be in equilibrium, or \( Q_c = Q_s = \text{const} \):

\[
\frac{1}{Q_c} \frac{dQ_c}{d\zeta} = \frac{1}{V} \frac{dV}{d\zeta} + \frac{1}{A} \frac{dA}{d\zeta} = 0
\]

For steady, uniform alluvial-channel flow that is turbulent and hydraulically rough, the following generalized form of resistance relationship is applied:

\[
V = c_{v} R S^{x}
\]

where \( c_{v} \) = coefficient related to the rough degree of sediments composing channel boundary; \( S \) = energy gradient of flow; and \( x \) and \( y \) = exponents varying with channel bed form or flow regime.
For fixed-bed or flatbed flow regime, the widely applied Manning-Strickler formula gives \( x \) and \( y \) values of 2/3 and 1/2, respectively. In terms of the study of Brownlie (1983), \( x \) and \( y \) have respective values of 0.5293 and 0.3888 for the lower flow regime and 0.6005 and 0.4605 for the upper flow regime. As a whole, \( x \) and \( y \) take the following values:

\[
x = 0.5293; \quad y = 0.3888; \quad \text{lower flow regime}
\]

\[
x = 0.6005; \quad y = 0.4605; \quad \text{upper flow regime}
\]

\[
x = \frac{1}{2}; \quad y = 0.667; \quad \text{fixed-bed or flatbed flow regime}
\]

For given \( S \) and sediment composition of channel boundary or \( c_v \), the variational form of Eq. (10) against \( \zeta \) is written as

\[
\frac{1}{V} \frac{dV}{d\zeta} = \frac{x}{R} \frac{dR}{d\zeta}
\]

As a result, the combination of Eqs. (8) and (12) yields

\[
\frac{1}{h} \frac{dh}{d\zeta} = \frac{\zeta + 2(1 + x)}{\zeta(\zeta + 2)(2 + x)}
\]

\[
\frac{1}{B} \frac{dB}{d\zeta} = \frac{(1 + x)\zeta + 2}{\zeta(\zeta + 2)(2 + x)}
\]

The sediment carrying capacity of flow has been illustrated with numerous approaches, among which the shear stress approach has gained the most popular applications. For this, this study applies the following bed-load transport formula in an attempt to cover bed-load transport conditions as widely as possible:

\[
q_{sc} = c_{e} \tau_{0}^{0} (\tau_{0} - \tau_{e})^{i}
\]

where \( q_{sc} \), \( c_{e} \), \( \tau_{0} \), \( \tau_{e} \), \( i \), and \( j \) = carrying capacity of flow for sediment discharge per unit channel width, a constant relating to sediment characteristics, the average shear stress, the critical shear stress for the incipient motion of sediments, and two exponents, respectively.

It is apparent that the generalized sediment transport formula in Eq. (14) covers a wide range of bed-load transport formulas that have been developed in terms of the shear stress approach. These include the DuBoys (1879) formula that gives \( i \) and \( j \) equal values of 1.0 and the Parker (1979) formula that suggests values of \(-3.0 \) for \( i \) and 4.5 for \( j \). In the special case of \( i = 0 \), the generalized formula in Eq. (14) also covers a wide range of available bed-load transport formulas. Typical examples are the Meyer-Peter–Müller (1948) formula that gives \( j \) a value of 1.5, the U.S. Waterways Experiment Station (1935) formula that suggests \( 1.5 < j < 1.8 \), and the O’Brien and Rindlaub (1934) formula that assigns 1.3 to 1.4 to \( j \).

From the generalized bed-load transport relationship in Eq. (14) and the relationship of \( Q_{sc} = q_{sc}B \), the following variational relationship can be derived:

\[
\frac{dQ_{sc}}{d\zeta} = Q_{sc} \left[ \frac{1}{B} \frac{dB}{d\zeta} + \left( \frac{i}{\tau_{0} - \tau_{e}} \right) \frac{d\tau_{0}}{d\zeta} \right]
\]

Because \( \tau_{0} = \gamma RS \) (\( \gamma \) = specific weight of water), the variation of \( \tau_{0} \) against \( \zeta \) can be found from Eq. (13) for a given \( S \) as

\[
\frac{1}{\tau_{0}} \frac{d\tau_{0}}{d\zeta} = \frac{1}{R} \frac{dR}{d\zeta} - \frac{2}{\zeta(\zeta + 2)} + \frac{1}{h} \frac{dh}{d\zeta} = \frac{2 - \zeta}{\zeta(\zeta + 2)(2 + x)}
\]

Incorporating the expressions of \( dB/d\zeta \) and \( d\tau_{0}/d\zeta \) in Eqs. (13) and (16) into Eq. (15) produces

\[
\frac{dQ_{sc}}{d\zeta} = Q_{sc} \left[ \frac{1}{B} \frac{dB}{d\zeta} + \left( \frac{i}{\tau_{0} - \tau_{e}} \right) \frac{d\tau_{0}}{d\zeta} \right]
\]

By letting \( dQ_{sc}/d\zeta = 0 \), the specific point of \( \zeta = \zeta_{e} \), at which alluvial channel flow achieves stationary equilibrium in terms of Eqs. (5) and (6), can be determined from the following relationship:

\[
\frac{\tau_{0} - \tau_{e}}{\tau_{e}} \bigg|_{\zeta = \zeta_{e}} = \frac{(1 + x - i)\zeta_{e} + 2(1 + i)}{(1 + x - i - j)\zeta_{e} + 2(1 + i + j)}
\]

which is equivalent to

\[
\frac{\tau_{0} - \tau_{e}}{\tau_{e}} \bigg|_{\zeta = \zeta_{e}} = \frac{j(\zeta_{e} - 2)}{(1 + x - i - j)\zeta_{e} + 2(1 + i + j)}
\]

It is noticeable in Eqs. (18) and (19) that when \( \tau_{0}/\tau_{e} = 1 \) or \( \tau_{0} - \tau_{e} = 0 \) for the incipient motion of bed sediment, \( \zeta = 2 \). When \( \tau_{0} - \tau_{e} > 0 \), \( \zeta > 2 \). This implies that the condition of \( \zeta = 2 \) defines the lower limit of channel form adjustment in stationary equilibrium. For general use, this limit is written as \( \zeta_{e}^{l} \). For a rectangular channel cross section, therefore, the flow resistance and bed-load transport formulas presented in Eqs. (10) and (14) yield

\[
\zeta_{e}^{l} = 2
\]

Because in natural streams \( \zeta \) is generally larger than \( \zeta_{e}^{l} \), the following conditions need to be satisfied in Eqs. (18) and (19):

\[
[\zeta_{e} > \zeta_{e}^{l}] \cap [(1 + x - i - j)\zeta_{e} + 2(1 + i + j) > 0]
\]

It is noticeable in Eq. (21) that there is a turning point when \( i + j = 1 + x \). This leads Eq. (21) to be expressed more specifically as

\[
\zeta_{e}^{l} < \zeta_{e} < (i + j)^{n}
\]

\[
\zeta_{e}^{l} \leq \zeta_{e} < (i + j)^{n}
\]

\[
\zeta_{e} \leq \zeta_{e} < (i + j)^{n}
\]

where \( (i + j)^{n} \) is defined as

\[
(i + j)^{n} = 1 + x
\]

Furthermore, the following extreme conditions can be identified from Eq. (22):

\[
\lim_{i + j \to -\infty} \left[ \frac{2(1 + i + j)}{1 + x - i - j} \right] = (\zeta_{e})^{-}
\]

\[
\lim_{i + j \to +\infty} \left[ \frac{2(1 + i + j)}{i + j - 1 - x} \right] = (\zeta_{e})^{+}
\]

where the superscripts – and + denote that the concerned functions in Eq. (24) vary from smaller and larger values, respectively, towards the lower limit \( \zeta_{e}^{l} \).

As a result, the following generalized condition emerges as the solution of Eq. (22):

\[
\zeta_{e}^{l} \leq \zeta_{e} < \zeta_{e}^{u}
\]

in which \( \zeta_{e}^{u} \) represents the upper limit of channel form adjustment in stationary equilibrium and can be determined from Eq. (22) as
Hence the condition under which flow achieves stationary equilibrium can become a simple linear relationship when \( i+j=(i+j)^* \). This linear relationship is the focus of this study and the following parts of this study present a detailed investigation of the conditions leading to it.

**Linearity in Relation to Bed-Load Transport**

By assigning \((i+j)^*\) to \(i+j\), the following simple linear relationship can be identified from the complicated nonlinear relationship in Eq. (18) or Eq. (19):

\[
\frac{\tau_0 - \tau_c}{\tau_c} = \frac{j}{2(2+x)} (\zeta_c - \zeta^*_e) \tag{28}
\]

where \(i\) and \(j\) are interrelated in the following form according to Eq. (23):

\[
i + j = (i+j)^* = 1 + x \tag{29}
\]

Letting \(\Delta \tau_0\) and \(\Delta \zeta_e\) be, respectively, the increments of \(\tau_0\) and \(\zeta_e\), Eq. (28) becomes

\[
\frac{\Delta \tau_0}{\tau_c} = \frac{j}{2(2+x)} \Delta \zeta_e \tag{30}
\]

In the special case of \(i=0\), thus \(j=1+x\), Eq. (30) becomes

\[
\frac{\Delta \tau_0}{\tau_c} = \frac{1+x}{2(2+x)} \Delta \zeta_e \tag{31}
\]

As analyzed earlier, the values of \(i\) and \(j\) vary within the wide range of \(0<i+j<(i+j)^*\) or \(i+j>(i+j)^*\) for the satisfaction of the general nonlinear condition of stationary equilibrium in Eq. (18) or Eq. (19). Nevertheless, the available bed-load transport formulas suggest that \(i\) and \(j\) take only certain values that tend to make Eqs. (18) and (19) become linear functions. In these formulas, the sum of \(i\) and \(j\) commonly falls right into or very close to the range of the variation of \((i+j)^*\) (1.5293–1.667). For example, the widely applied Parker (1979) formula gives a value of 1.5 to the term of \(i+j\) \((i=-3.0\) and \(j=4.5\), which is very close to the smallest value of \((i+j)^*\), 1.5293, for the lower flow regime. In contrast, the DuBoys (1879) formula assigns 2.0 to the term of \(i+j\) \((i=j=1.0\), which falls considerably away from the range of the variation of \((i+j)^*\). Perhaps because of this, the DuBoys formula has been less commonly applied and only regarded as a classic one. In the special case of \(i=0\), the commonly applied Meyer-Peter–Müller (1948) and the U.S. Waterways Experiment Station (1935) formulas assign \(j=1.5\) and \(1.5<j<1.8\), respectively, which largely fall into the range of the variation of \((i+j)^*\). However, the O’Brien and Rindlaub (1934) formula provides \(j\) a value of 1.3 to 1.4, which falls considerably away from the range of the variation of \((i+j)^*\). This may provide an explanation as to why the formula has been less frequently applied.

In the special case of \(\Delta \tau_0=0\), it is noticeable in the linear condition of stationary equilibrium in Eq. (28) that \(\zeta_e=\zeta^*_e\). This is also the condition under which flow achieves stationary equilibrium in a frictional fixed-boundary open channel. Hence the condition for flow in a fixed-boundary open channel illustrates only a special case of the linear condition for alluvial channel flow. This is because flow either in alluvium or in a fixed-boundary channel needs to spend energy to overcome boundary resistance. Hence the linear condition in Eq. (28) may not be applicable to flow in an open channel in which energy is expended dominantly in the forms of potential and kinetic energy, such as flow in frictionless fixed-boundary channels or in steep mountain streams.
Linearity in Relation to River Channel Geometry

Besides the evidence from the values of \( i \) and \( j \) adopted in the commonly applied bed-load transport formulas, two criteria that can be drawn from observed river channel geometry provide additional evidence to the existence of the linear behavior for flow in alluvium. The first is the maximum value of width/depth ratio that occurs in natural rivers. Although detailed information on the true value of the maximum is lacking at this stage, observations suggest that the value can be up to 100. By assigning this value as the upper limit of \( \xi_e \) or \( \xi_e' \), it can be calculated from Eq. (24) that to make \( \xi_e \) vary from 2 up to 100 in the situation of \( i+j > (i+j)^* \), \( i+j \) needs to take values of 1.58, 1.65, and 1.72 for the lower, upper, and flat-bed flow regimes, respectively. In comparison with the values of \( (i+j)^* \) (1.5239, 1.6005, and 1.65 for the lower, upper, and flat-bed flow regimes, respectively), these values are only slightly larger in the corresponding flow regimes. Hence the criterion that the maximum value of the width/depth ratio of river channels reaches 100 can be satisfied approximately with the condition under which flow behaves in a linear form, or \( i+j = (i+j)^* \).

The second criterion is the theoretical equilibrium channel geometry that can be derived from the general nonlinear condition under which alluvial channel flow achieves stationary equilibrium in Eq. (18) or Eq. (19). Because the condition is a function of \( i+j \), a comparison of the theoretical results derived from the condition with empirically developed hydraulic geometry relations can directly reveal what an appropriate value \( i+j \) can adopt. As demonstrated in the following, this comparison is very effective in the situation of \( 0 < i+j \leq (i+j)^* \). In the situation of \( i+j > (i+j)^* \), the comparison becomes less effective because as stated earlier, the values of \( i+j \) for the satisfaction of \( 2 \leq \xi_e \leq 100 \) are very close to \( (i+j)^* \).

As is well-known, hydraulic geometry relations are determined by multiple variables. Nevertheless, they appear most commonly in the following forms when the effects of factors other than flow discharge \( Q \), typically bank strength, channel roughness, and channel slope, are all taken as constants (e.g., Park 1977; Rhodes 1987; Hey and Thorne 1986; Huang and Nanson 1995; Huang and Warner 1995; Julien and Wargadalam 1995; Huang 1996; Huang and Nanson 1997, 1998)

\[
B \propto Q^{0.5} \\
h \propto Q^{0.3} \\
V \propto Q^{0.2}
\] (32)

In order to obtain theoretical equilibrium channel geometry that is comparable with Eq. (32), we consider only the special case of \( i=0 \) in the following mathematical analysis of the complicated nonlinear form of Eq. (18). By doing so, a much more complicated mathematical derivation of equilibrium channel geometry is avoided. This simplification, together with the relationships of \( \tau_e = \gamma R S \) and \( R = h [\xi_e (\xi+2)] \), leads the following theoretical equilibrium channel geometry relationship to be derived from Eq. (18):

\[
h = \frac{\tau_e}{\gamma S} \cdot \frac{(1+x)\xi_e + 2}{(1+x-j)\xi_e + 2(1+j)} \cdot \frac{\xi_e + 2}{\xi_e} \] (33)

For maintaining flow continuity, or \( Q = Q_c = VA \), and flow resistance law illustrated in Eq. (10), equilibrium channels need to satisfy the following relationship when the relationships of \( A = \xi h^2 \) and \( R = h [\xi/(\xi+2)] \) are incorporated into Eq. (10):

\[
\frac{Q}{c_v S} = h^{2+x} \cdot \frac{\xi_e^{1+x}}{(\xi_e + 2)^1} \] (34)

By eliminating \( h \) from Eqs. (33) and (34), equilibrium channel shape \( \xi_e \) can be determined from the following relationship:

\[
\frac{V^{2+x}}{c_v T_v^{2+x}} Q S^{2+x-y} = f_1(\xi_e) \] (35)

in which

\[
f_1(\xi_e) = \left[ \frac{(1+x)\xi_e + 2}{(1+x-j)\xi_e + 2(1+j)} \right] \frac{2+x}{\xi_e} \frac{(\xi_e + 2)^2}{\xi_e} \] (36)

It is clear that it is not possible to find solutions of \( f_1(\xi_e) \) without making simplifications. For this, \( f_1(\xi_e) \) is assumed to take simply the following power function:

\[
\xi_e^3 = f_1(\xi_e) \] (37)

This leads Eq. (35) to produce the following power-form relationship when only the effect of flow discharge \( Q \) is concerned:

\[
\xi_e^3 \propto Q \] (38)

In the same form of approximation, the power-form relationship of \( h = h(Q, \xi_e) \) can be found from Eq. (34) as

\[
h \propto Q^{1/(2+x)\alpha_2/\alpha_1} \] (39)

As a consequence, theoretical equilibrium channel geometry can be written into the following commonly applied hydraulic geometry relations:

\[
B \propto Q^b \\
h \propto Q^f \\
V \propto Q^m
\] (40)

where exponents \( b \), \( f \), and \( m \) are determined by the following relationships:

\[
b = \frac{1}{2+x^2} + \frac{1 + \alpha_2}{\alpha_1} \] (41)

\[
f = \frac{1}{2+x^2} + \frac{\alpha_2}{\alpha_1} \]

\[
m = 1 - b - f
\]

Although \( \alpha_2 \) is a function of \( \xi_e \) only as shown in Eq. (39), \( \alpha_1 \) is a function of both \( \xi_e \) and \( j \) as demonstrated in Eqs. (36) and (37). In order to obtain the averaged values of \( \alpha_1 \) and \( \alpha_2 \) over the range of \( 2 \leq \xi_e \leq 100 \) at a high degree of accuracy, \( \xi_e \) is assigned with a series of values that start from 2 and then increase at increments of 1 until they reach 100. Consequently, \( j \) is assigned with values that vary from 0.5 up to its upper limits \( \xi_e' \). Under the condition of \( 2 \leq \xi_e \leq 100 \), it can be determined from Eq. (26) that these upper limits take values of 1.58, 1.65, and 1.72 for the lower, upper, and flatbed flow regimes, respectively. They are slightly larger than \( 1+x \) or \( (i+j)^* \) for the corresponding flow re-
found that statistically satisfactory values of regression analyses are undertaken with Microsoft Excel. It is shown that monotonic variations with flow indeed tend to exhibit a linear characteristic when it occurs in general terms that alluvial channel equilibrium channel geometry relations. Despite this slight inconsistency, it appears in general terms that the theoretical flow resistance relationships and power-function approximations for complex equilibrium channel geometry relations. The results of this study are based on both semitheoretical flow resistance relationships and power-function approximations for complex equilibrium channel geometry relations. The values of $b$, $f$, and $m$ obtained in terms of Eq. (41) and the values of $\alpha_1$ and $\alpha_2$ presented in Table 1 are plotted against the corresponding values of $j$ in Fig. 2. It is noticeable in the figure that with an increase in $j$ from 0.5 to its upper limits determined by the condition of $2 \leq \zeta_0 = 100$, $b$ progressively decreases from a value greater than 1.0 downwards to about 0.5. In contrast, $f$ and $m$ progressively increase from values that are either negative or just above zero upwards to about 0.3 and about 0.2, respectively. In the special case of $j=(i+j)^* = 1+x$, at which the general nonlinear condition of stationary equilibrium in Eq. (18) or Eq. (19) becomes a simple linear function, the averaged values of $b$, $f$, and $m$ are very close to field observations. However, the obtained averaged values of $b$, $f$, and $m$ can get even closer to field observations when $j$ is slightly larger than $1+x$ but less than its upper limits. The cause for that the closest values of $b$, $f$, and $m$ do not occur exactly at $j=1+x$ or $j=(i+j)^*$ for $i=0$ may be because: (1) field observations are statistical results; and (2) the theoretical results of this study are based on both semitheoretical flow resistance relationships and power-function approximations for complex equilibrium channel geometry relations. Despite this slight inconsistency, it appears in general terms that alluvial channel flow indeed tends to exhibit a linear characteristic when it achieves stationary equilibrium.

Table 1. Fitted Values of $\alpha_1$ and $\alpha_2$ in Eqs. (37) and (39)

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\alpha_1$</th>
<th>$r^2$</th>
<th>$\alpha_1$</th>
<th>$r^2$</th>
<th>$\alpha_1$</th>
<th>$r^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.9184</td>
<td>0.9991</td>
<td>0.9773</td>
<td>0.999</td>
<td>0.9739</td>
<td>0.9989</td>
</tr>
<tr>
<td>1.0</td>
<td>1.197</td>
<td>0.994</td>
<td>1.1739</td>
<td>0.9949</td>
<td>1.1561</td>
<td>0.9956</td>
</tr>
<tr>
<td>1.2</td>
<td>1.3845</td>
<td>0.986</td>
<td>1.3317</td>
<td>0.9874</td>
<td>1.2937</td>
<td>0.9887</td>
</tr>
<tr>
<td>1.4</td>
<td>1.8217</td>
<td>0.9811</td>
<td>1.6458</td>
<td>0.9798</td>
<td>1.542</td>
<td>0.9802</td>
</tr>
<tr>
<td>1.5</td>
<td>2.551</td>
<td>0.9938</td>
<td>1.9982</td>
<td>0.9817</td>
<td>1.7761</td>
<td>0.9783</td>
</tr>
<tr>
<td>1.5293</td>
<td>3.301</td>
<td>0.9967</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.55</td>
<td>4.1196</td>
<td>0.9198</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.58</td>
<td>5.0252</td>
<td>0.8786</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td></td>
<td>2.2638</td>
<td>0.9853</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.6005</td>
<td>3.3735</td>
<td>0.9968</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.65</td>
<td></td>
<td>4.821</td>
<td>0.9205</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5/3</td>
<td>3.2688</td>
<td>0.9958</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.7</td>
<td>4.3811</td>
<td>0.9069</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.72</td>
<td>5.2489</td>
<td>0.8817</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\alpha_2 & \quad r^2 \\
-0.4081 & -0.3968 -0.3992 -0.3902 -0.9999
\]

Fig. 2. Variations of $b$, $f$, and $m$ with $i+j$

Physical Implications of $H$

In terms of the linear characteristic of alluvial channel flow presented in Eqs. (28)–(31), a nondimensional number $H$ can be defined as

\[
H = \frac{\Delta \zeta / \zeta_i}{\Delta \zeta_j}
\]  

(42)

As a result, alluvial channel flow in equilibrium includes the following three cases:

\[
\begin{align*}
H & > k \\
H & = k \\
H & < k
\end{align*}
\]  

(43)

where $k$ can be determined from the following relationship according to Eq. (30):
As stated earlier, the value of $j$ in Eq. (44) is subject to the condition of $j=1+x$ for the generalized sediment transport formula in Eq. (14). In the special case of $i=0$, $j=1+x$ and so $k$ can be determined as

$$k = \frac{j}{2(2+x)}$$  

(44)

It is interesting to note in Eq. (45) that for different flow regimes, $k$ varies only slightly and a constant of 0.3 appears acceptable for transporting sediment load. As noted in the generalized sediment transport formula of Eq. (14) and the relationship of $Q_e=Q_fR$, the sediment transporting capacity $Q_e$, is a function of both channel width $B$ and flow shear stress $\tau_0$ for a given sediment size. Nevertheless, there exists a quasi-parabolic (rising and falling) relationship between average flow shear stress $\tau_0$ and channel shape factor $\zeta$ in Eq. (16), while a monotonic relationship maintains between channel width $B$ and $\zeta$ in Eq. (12). As a result, $\tau_0$ plays a major role in transporting sediment load in a narrow and deep channel, quantitatively resulting in $H/k$. While in a wide and shallow channel, the role of $B$ overweights that of $\tau_0$, yielding $H<k$. Only in the ideal case where both the roles of $\tau_0$ and $B$ are equally significant within a channel that is neither very wide and shallow nor very narrow and deep does $H=k$ occur.

Importantly, the existence of $H$ demonstrates that geometrically distorted models are inevitable when modeling sediment (bed-load) transport in alluvial channels. In the construction of such a distorted model, nevertheless, the value of $H$ is a scale independent number and so needs to be kept unchanged in order to achieve similarity between a prototype and the model. This helps to clarify the confusion over what similarity criterion needs to be imposed in modeling alluvial channel flow. For a long time, the following nondimensional empirical criterion has been adopted for the same prototype bed material (e.g., Einstein and Chen 1956; Bogárdi 1974):}

$$\frac{d_f}{h_fS_f} = 1 \quad \text{or} \quad \frac{d_f}{KRS_f} = 1$$  

(46)

where $d_f$, $h_f$, $R$, and $S_f$ represent the scaled forms of sediment size $d$, flow depth $h$, hydraulic radius $R$, and channel slope $S$, respectively. In a comparison with the definition of $H$ in Eq. (43), it can be noted that this criterion considers the scaling effect of shear stress only and ignores that of channel shape on sediment transport.

Recently, Griffiths (2003) suggested the following criterion for constructing a distorted river model:

$$\frac{B_r}{h_r} \cdot S_r = 1$$  

(47)

where $B_r$ is scaled form of channel width $B$. Although this criterion is different to the scale independent number $H$ defined in Eq. (42) in our study, it considers the scaling effect of channel shape and yields scaling relations that are consistent with empirical hydraul-
The significant points for this number are as follows: (1) it illustrates a hydraulic similarity criterion for modeling sediment (bedload) transport in alluvial channels; and (2) it can be used to assess whether alluvial channel flow uses energy in the most efficient way or is at the state of stationary equilibrium.

Finally, this study indicates that the recognition of the linear characteristic of alluvial channel flow might lead to more fruitful results. The first is the development of a universally applied bedload transport formula. Since the linear condition explains why some of the available sediment transport formulas have gained more popular use than the others, it thus means that sediment transport formulas can be developed without so much dependence on flume and field data. Our ongoing study is investigating the possibility for developing a universally applied sediment transport formula by implementing the condition as a criterion into data analysis.

The second result is the development of a variational approach for understanding the physical mechanism behind the linear behavior of flow in more complex environments. These include the linear relationships of the spacing distance of successive pools or riffles with channel width in riffle-pool systems (e.g., Keller and Melhorn 1978), the step steepness with average bed slope in step-pool systems (e.g., Abrahams et al. 1995), and the wavelength of meanders with channel width in meandering systems (e.g., Dury 1955; Williams 1986). Theoretically speaking, these significantly different forms of linear characteristics are possibly the products of the achievement of stationary equilibrium by flow under more complex environmental conditions. Therefore they might be explainable by incorporating the restrictions into either the mathematical analytical approach advocated by Kirkby (1977) and Huang and co-workers (e.g., Huang and Nanson 2000, 2001, 2002; Huang et al. 2002, 2004a,b) or the computation based analytical approach implemented by Chang (1979a,b, 1980a,b); White et al. (1982); Bettess and White (1983); Chang (1988); Wang and Zhang (1989); Millar and Quick (1993, 1998); Millar (2000); and Eaton et al. 2004. To validate this inference, further detailed studies are under way.

Notation

The following symbols are used in this paper:

\[ h = \text{flow depth or mean channel depth} \]
\[ h_s = \text{scaled form of channel depth} \]
\[ i = \text{exponent of shear stress in the sediment transport relationship of Eq. (14)} \]
\[ j = \text{exponent of the increment of shear stress in the sediment transport relationship of Eq. (14)} \]
\[ (i+j)^* = \text{value of the sum of } i \text{ and } j \text{ leading to the linear relationship between } \tau_0 \text{ and } \zeta; \]
\[ k = \text{value (0.3) of } H \text{ when flow is at the most efficient state for transporting sediment load}; \]
\[ m = \text{exponent of flow discharge for flow velocity in hydraulic geometry relations}; \]
\[ p = \text{constant}; \]
\[ Q = \text{water discharge imposed to flow}; \]
\[ Q_c = \text{carrying capacity of flow for water discharge}; \]
\[ Q_s = \text{sediment discharge imposed to flow}; \]
\[ Q_{sc} = \text{carrying capacity of flow for sediment discharge}; \]
\[ (Q_{sc})_{max} = \text{maximum carrying capacity of flow for sediment discharge}; \]
\[ q_{sc} = \text{carrying capacity of flow for sediment discharge per unit channel width}; \]
\[ R = \text{hydraulic radius of channel}; \]
\[ R_s = \text{scaled form of hydraulic radius} \]
\[ r^2 = \text{square of the correlation coefficient}; \]
\[ S = \text{energy slope of flow}; \]
\[ S_r = \text{scaled form of energy slope of flow} \]
\[ V = \text{average flow velocity}; \]
\[ x = \text{exponent of hydraulic radius in the flow resistance relationship of Eq. (10)} \]
\[ y = \text{exponent of energy slope in the flow resistance relationship of Eq. (10)} \]
\[ \alpha_1, \alpha_2 = \text{exponents of } \zeta; \]
\[ \gamma = \text{specific weight of water}; \]
\[ \Delta \tau_0 = \text{increment of } \tau_0 \text{ or } \tau_0 - \tau; \]
\[ \Delta \zeta_e = \text{increment of } \zeta_e \text{ or } \zeta_e - \zeta_e^*; \]
\[ \zeta_e = \text{channel shape factor defined as width/depth ratio or } B/h; \]
\[ \xi_e = \text{width/depth ratio of stationary equilibrium channel}; \]
\[ \xi_e^l = \text{lower limit of the variation of } \zeta_e \text{ and for a rectangular section, } \xi_e^l = 2; \]
\[ \zeta_e^u = \text{upper limit of the variation of } \zeta_e; \]
\[ \tau_c = \text{critical shear stress for the incipient motion of sediments}; \]
\[ \tau_0 = \text{shear stress of flow, or } \gamma R S. \]

References


U.S. Waterways Experiment Station. (1935). *Studies of river bed materials and their movement, with special reference to the lower Mississippi River*, USWES Paper 17, Vicksburg, Miss.


